2023-24 MATH2048: Honours Linear Algebra II

Homework 4

Due: 2023-10-09 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date. All questions are selected from Friedberg §2.4-2.5.

1. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible.

   Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible. (By this, the book means when A, B are not square, AB can still be invertible. No need to do this part.)

2. Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(F) \to M_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

3. Let A be an $n \times n$ matrix.

   (a) Suppose that $A^2 = 0$. Prove that A is not invertible.

   (b) Suppose that $AB = 0$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

   We select the following questions from Artin's Algebra (chap 4) in place of original Q4-Q5.

4. Let A and B be 2×2 matrices. Determine the matrix of the operator $T(M) = AMB$ on the space $F^{2\times 2}$ of 2×2 matrices, with respect to the basis $(e_{11}, e_{12}, e_{21}, e_{22})$ of $F^{2\times 2}$.

5. Find all real 2×2 matrices that carry the line $y = x$ to the line $y = 3x$.

   The following are extra recommended exercises not included in homework.. Qa-Qb were original compulsory Q4-Q5, and will appear in next homework. Keep your answer till then if you have done these.

(a) (Submit in HW5) For each of the following pairs of ordered bases β and β' for $P_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

(a) $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

(b) $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$

(b) (Submit in HW5) For each matrix A and ordered basis β, find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

(a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

(b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

6. Prove that "is similar to" is an equivalence relation on $M_{n\times n}(F)$.

7. Prove that if A and B are similar $n \times n$ matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

8. In R^2, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

   (a) T is the reflection of R^2 about L.

   (b) T is the projection on L along the line perpendicular to L. (See the definition of projection in the exercises of Section 2.1.)

9. Let V and W be finite-dimensional vector spaces and $T : V \to W$ be an isomorphism. Let V_0 be a subspace of V.

   (a) Prove that $T(V_0)$ is a subspace of W.

   (b) Prove that $\dim(V_0) = \dim(T(V_0))$.

10. For each of the following linear transformations T, determine whether T is invertible and justify your answer.

   (a) $T : R^2 \to R^3$ defined by $T(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.

   (b) $T : R^2 \to R^3$ defined by $T(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$.

   (c) $T : R^3 \to R^3$ defined by $T(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$.